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Collective Field Description of
Spin Calogero-Sutherland Models

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Abstract

Using the collective field technique, we give the description of the spin Calogero-Sutherland Model (CSM) in terms of free bosons. This approach can be applicable for arbitrary coupling constant and provides the bosonized Hamiltonian of the spin CSM. The boson Fock space can be identified with the Hilbert space of the spin CSM in the large N limit. We show that the eigenstates corresponding to the Young diagram with a single row or column are represented by the vertex operators. We also derive a dual description of the Hamiltonian and comment on the construction of the general eigenstates.

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1   Introduction

The Calogero-Sutherland model (CSM) [1, 2] has been an interesting laboratory to study the fractional statistics in $(1+1)$-dimension [3, 4, 5]. Its paradigmatic rôle as the anyonic analog of the free boson or fermion gas has been established. Also, the CSM is related to various branches of physics and contains many interesting aspects in mathematical physics [6]. Especially, it is known that this model is the universal Hamiltonian for the disordered systems [7].

Many variants of the CSM now exist, for example, its lattice cousin, the so-called Haldane-Shastry model [8], and multicomponent version, the *spin* (or dynamical) CSM [9, 10, 11, 12]. A lot of intriguing results have been obtained in connection with these models where the Yangian symmetry [13, 14, 15, 16] plays essential rôle to explain the degeneracy of the spectrum. For particular couplings, $\alpha = 2, 1/2$, this nonlinear symmetry is known to be realized through the spinon basis (or the vertex operators of the free boson) [17, 18, 19]. This is the point where the symmetry of the system is enhanced to the level one $su(2)$ Kac-Moody algebra.

In our previous studies [20], the bosonization for the CSM has been given (see for the related works [21, 22, 23, 24]). One of the essential observations in those works was that the collective coordinate description of the system is equivalent to the Coulomb gas description of the minimal model of conformal field theory. In particular, two screening currents of the minimal model are naturally identified with the generating functionals of one particle and one hole states. Similarly, any eigenstate (which is known as the Jack polynomial) can be identified with the singular vector of the appropriate W algebra.

In this letter, we show that some part of the above scenario can be generalized to the spin CSM without any restriction on the coupling constant. We describe the Hamiltonian in terms of multicomponent free bosons. In our method, the correspondence between the spin CSM Hilbert space and the free boson Fock space is one to one. We explicitly obtain the generating functional of one particle (hole) excited states as vertex operator. General eigenstates would be written as the product of the vertex operators. We also derive the "dual" Hamiltonian defined by the action of the original Hamiltonian on such states. The integrability of this dual Hamiltonian directly follows from its construction.

There are, however, some differences from the spinless CSM. For example, the duality (or the charge conjugation) symmetry of the system disappears. Therefore, it becomes rather difficult to relate the Hamiltonian with the loop algebra such as the Virasoro algebra or the Kac-Moody algebra. In particular, it is still hard to see the connection with the Yangian or the Kac-Moody symmetry even if we pick $\alpha = 2$ or $1/2$.

In the conclusion, we comment how one can construct the general eigenstates of the spin CSM by using the dual Hamiltonian.

2 Collective Field Description of Spin CSM

Let us write down the reduced form of the Hamiltonian for the spin CSM (see [10, 14]). Performing a "gauge" transformation, the Hamiltonian is given by

$$\mathcal{H} = \alpha \sum_{i=1}^{N} D_{x_i}^2 + \sum_{i<j} \frac{x_i + x_j}{x_i - x_j}(D_{x_i} - D_{x_j}) - 2\sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2}(1 - K_{ij}), \qquad (1)$$

where $\alpha \in \mathbf{C}$ is the coupling constant, $D_x \equiv x\frac{\partial}{\partial x}$ and K_{ij} is the (coordinate) exchange operator, namely, for a function f in x_i's,

$$K_{ij}f(\cdots, x_i, \cdots, x_j, \cdots) \equiv f(\cdots, x_j, \cdots, x_i, \cdots).$$

The wave function of the Hamiltonian is described by the coordinates x_i and the spin variables σ_i attached to them. Each spin variable takes values in the set $\{1, 2, \cdots, s\}$ (*i.e.*, we consider the s-component system) and the wave function should be invariant under the simultaneous exchange of both variables,

$$\psi(\cdots, x_i\sigma_i, \cdots, x_j\sigma_j, \cdots) = \psi(\cdots, x_j\sigma_j \cdots, x_i\sigma_i, \cdots). \qquad (2)$$

In other words, the exchange of the coordinate and that of the spin variable have same effect when they are acted on the wave function.

One of the nontrivial properties of the spin CSM Hamiltonian is that, when we try to diagonalize the Hamiltonian, we are able to restrict the Hilbert space such that the spin variable for each particle is fixed. More precisely, let us denote $x_i^{(\sigma)}$'s as those coordinates whose spin variables take value $\sigma \in \{1, \cdots, s\}$. Then the restricted Hilbert space is defined by the set of functions which are symmetric under the exchange $x_i^{(\sigma)}$ and $x_j^{(\sigma)}$ for each spin σ. At first glance such restriction may not be compatible with the action of the Hamiltonian because of the terms which include the exchange operator K_{ij}. These terms exchange the x coordinates alone and leave the spin variables untouched. However, we can prove by an explicit computation that the undesirable terms vanish.

In the restricted Hilbert space, we can apply the standard bosonization (or collective coordinates) technique. Let us define the power sum for each spin,

$$p_n^{(\sigma)} \equiv \sum_{i=1}^{N^{(\sigma)}} (x_i^{(\sigma)})^n, \qquad (3)$$

where $N^{(\sigma)}$ is the number of particles with spin σ. We also introduce free bosons $a_n^{(\sigma)}$, $n \in \mathbf{Z}$, and boson fields $\phi_\pm^{(\sigma)}(\xi)$, $(\sigma = 1, \cdots, s)$, such that,

$$[a_n^{(\sigma)}, a_m^{(\sigma')}] = n\delta^{\sigma,\sigma'}\delta_{n+m,0}, \qquad \phi_\pm^{(\sigma)}(\xi) = \mp \sum_{n>0} \frac{1}{n} a_{\pm n}^{(\sigma)} \xi^{\mp n}. \qquad (4)$$

The bosonization method is to replace $p_n^{(\sigma)}$ by the free boson creation operator $a_{-n}^{(\sigma)}$. The collective coordinate description becomes exact in the limit that the number of the particles, *i.e.*, $N^{(\sigma)}$'s become all infinite. The replacement $p_n^{(\sigma)} \leftrightarrow a_{-n}^{(\sigma)}$ can be systematically carried out by introducing the operator $\langle \mathcal{V} |$,

$$\langle \mathcal{V} | \equiv \langle N | \exp \left\{ \sum_\sigma \sum_{n>0} \frac{1}{n} p_n^{(\sigma)} a_n^{(\sigma)} \right\}, \tag{5}$$

with the lowest weight state $\langle N |$ such that $\langle N | a_{-n}^{(\sigma)} = 0$, $n > 0$ and $\langle N | a_0^{(\sigma)} = N^{(\sigma)} \langle N |$. Taking the inner product with this bra state, we can translate the Fock space of free bosons into the restricted Hilbert space of the spin CSM. Namely, $\langle \mathcal{V} |$ translates coordinates $x^{(\sigma)}$ to bosons $a_n^{(\sigma)}$ as follows,

$$p_n^{(\sigma)} \langle \mathcal{V} | = \langle \mathcal{V} | a_{-n}^{(\sigma)}, \qquad n \frac{\partial}{\partial p_n^{(\sigma)}} \langle \mathcal{V} | = \langle \mathcal{V} | a_n^{(\sigma)}. \tag{6}$$

In the limit $N^{(\sigma)} \to \infty$, this correspondence is one to one. In other words, any operator which acts on the restricted Hilbert space can be rewritten by free boson oscillators. In particular, the Hamiltonian is bosonized as follows. Firstly, we shall decompose the Hamiltonian (1) into two parts,

$$\mathcal{H}(x) = \sum_{\sigma=1}^s \mathcal{H}^{(\sigma)}(x^{(\sigma)}) + \sum_{\sigma<\sigma'} \mathcal{H}_{int}^{(\sigma\sigma')}(x^{(\sigma)}, x^{(\sigma')}), \tag{7}$$

with

$$\mathcal{H}^{(\sigma)} = \alpha \sum_{i=1}^{N^{(\sigma)}} \left(D_{x_i^{(\sigma)}} \right)^2 + \sum_{i<j} \frac{x_i^{(\sigma)} + x_j^{(\sigma)}}{x_i^{(\sigma)} - x_j^{(\sigma)}} (D_{x_i^{(\sigma)}} - D_{x_j^{(\sigma)}}), \tag{8}$$

$$\mathcal{H}_{int}^{(\sigma\sigma')} = \sum_{i,j} \frac{x_i^{(\sigma)} + x_j^{(\sigma')}}{x_i^{(\sigma)} - x_j^{(\sigma')}} (D_{x_i^{(\sigma)}} - D_{x_j^{(\sigma')}}) - 2 \sum_{i,j} \frac{x_i^{(\sigma)} x_j^{(\sigma')}}{(x_i^{(\sigma)} - x_j^{(\sigma')})^2} (1 - K_{x_i^{(\sigma)}, x_j^{(\sigma')}}). \tag{9}$$

Then, the bosonized Hamiltonian $\hat{\mathcal{H}} = \sum_\sigma \hat{\mathcal{H}}^{(\sigma)} + \sum_{\sigma<\sigma'} \hat{\mathcal{H}}_{int}^{(\sigma\sigma')}$, where $\mathcal{H} \langle \mathcal{V} | = \langle \mathcal{V} | \hat{\mathcal{H}}$, is given by the formulae,

$$\hat{\mathcal{H}}^{(\sigma)} = \sum_{n,m>0} \left(a_{-n}^{(\sigma)} a_{-m}^{(\sigma)} a_{n+m}^{(\sigma)} + \alpha a_{-n-m}^{(\sigma)} a_n^{(\sigma)} a_m^{(\sigma)} \right) + \sum_{n>0} \left(\alpha n - n + a_0^{(\sigma)} \right) a_{-n}^{(\sigma)} a_n^{(\sigma)}, \tag{10}$$

$$\hat{\mathcal{H}}_{int}^{(\sigma\sigma')} = \sum_{n,m>0} a_{-n}^{(\sigma)} a_{-m}^{(\sigma')} \left(a_{n+m}^{(\sigma)} + a_{n+m}^{(\sigma')} \right) + \sum_{n>0} \left(a_0^{(\sigma')} a_{-n}^{(\sigma)} a_n^{(\sigma')} + a_0^{(\sigma)} a_{-n}^{(\sigma')} a_n^{(\sigma)} \right)$$
$$+ \oint \frac{d\xi \, d\eta}{\xi \, \eta} \sum_{n,m\geq 0} \xi^n \eta^m a_{-n}^{(\sigma)} a_{-m}^{(\sigma')} e^{\sum_{n>0} \frac{1}{n} (\xi^{-n} - \eta^{-n})(a_n^{(\sigma')} - a_n^{(\sigma)})} \sum_{k>0} k \left(\frac{\xi^k}{\eta^k} + \frac{\eta^k}{\xi^k} \right) \tag{11}$$

Here $\oint \frac{dx}{x} f(x)$ stands for the constant term of $f(x)$. The proof is similar to that in our previous papers [20]. The essential point is that $\mathcal{H}_{int}^{(\sigma\sigma')} \langle \mathcal{V} |$ has no pole at $x_i^{(\sigma)} = x_j^{(\sigma')}$

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and is a power series in $x_i^{(\sigma)}$ and $x_j^{(\sigma')}$'s. To treat the parts which include the exchange operators, we used

$$
\begin{aligned}
K_{x_i^{(\sigma)} x_j^{(\sigma')}} \langle \mathcal{V}| &= \langle \mathcal{V}| \, e^{\sum_{n>0} \frac{1}{n} \left((x_i^{(\sigma)})^n - (x_j^{(\sigma')})^n \right) (a_n^{(\sigma')} - a_n^{(\sigma)})} \\
&= \langle \mathcal{V}| \oint \frac{d\xi}{\xi} \frac{d\eta}{\eta} \sum_{n,m \geq 0} \xi^n \eta^m (x_i^{(\sigma)})^n (x_j^{(\sigma')})^m \, e^{\sum_{n>0} \frac{1}{n} (\xi^{-n} - \eta^{-n})(a_n^{(\sigma')} - a_n^{(\sigma)})}. \quad (12)
\end{aligned}
$$

In the Appendix, we will give examples of the eigenstates of this bosonized Hamiltonian for the low degree cases.

Remark that the third term of $\hat{\mathcal{H}}_{int}^{(\sigma\sigma')}$ is rewritten by using boson fields $\phi_+^{(\sigma)}(\xi)$ and $D_\xi \phi_{\leq 0}^{(\sigma)}(\xi) \equiv \sum_{n>0} a_{-n}^{(\sigma)} \xi^n + a_0^{(\sigma)}$ as follows,

$$
\oint_{|\eta|>|\xi|} \frac{d\eta d\xi}{(\eta - \xi)^2} : D_\xi \phi_{\leq 0}^{(\sigma)}(\xi) e^{\phi_+^{(\sigma)}(\xi) - \phi_+^{(\sigma')}(\xi)} D_\eta \phi_{\leq 0}^{(\sigma')}(\eta) e^{-\phi_+^{(\sigma)}(\eta) + \phi_+^{(\sigma')}(\eta)} : . \quad (13)
$$

Here $: * :$ is the usual normal ordering.

3 One Particle (Hole) States and Vertex Operators

In this section, we will show that the wave function of the one particle (hole) excited states can be expressed as the vertex operator of the free bosons. Before proceeding to the explanation, it may be better to illustrate the characterization of each eigenstate. As it is well-known, the eigenstates of the CSM without spin degrees of freedom can be indexed by the Young diagrams. Each row (column) in the diagram corresponds to the particle (hole) excitations of the CSM (see, for example, [5]). For each diagram, there is only one eigenstate, and the eigenvalue is determined from the diagram.

Even if we introduce the spin degrees of freedom, most of the structure remains the same. The eigenstates are again indexed by the Young diagrams. The eigenvalue is also determined by the diagram and it is actually the same as the spinless case. The difference, however, is that the eigenstate is not unique for each diagram, $i.e.$, the spectrum is degenerate. This is caused by the existence of the Yangian symmetry [13, 14].

There is a simple method to count the degeneracy of states for each Young diagram. With s colors that we have, we paint each box of the diagram according to the rule: the boxes in the same row have the same color and there is no constraint for the colors in the each column. The colored Young diagram after this prescription is indexed as $(\lambda_1 \sigma_1, \lambda_2 \sigma_2, \cdots, \lambda_N \sigma_N)$ where $\lambda_i \in \mathbf{Z}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ and $\sigma_i \in \{1, \cdots, s\}$. We identify the diagrams which can be obtained from one another by permuting the colors for each row with the same length.

This prescription is an obvious consequence of the fact that the number of the boxes for each row can be identified with the momentum of a quasi-particle. Since it has a

color, we need to paint each row by the same color. On the other hand, the number of boxes of each column is identified with the momentum of a quasi-hole. The colors which appear on each column can be identified with the colors of a quasi-particle which occupies the upper levels. Let us illustrate it in Fig.1. For simplicity we pick $\alpha = 1$ and consider the state depicted in Fig.1A. There are three particles written as a, b and c with spin 1, 2 and 1, respectively, and four holes x, y, z and w. This state can be rewritten as the Young diagram in Fig.1B. We see that particles are mapped to the rows and holes to the columns, respectively.
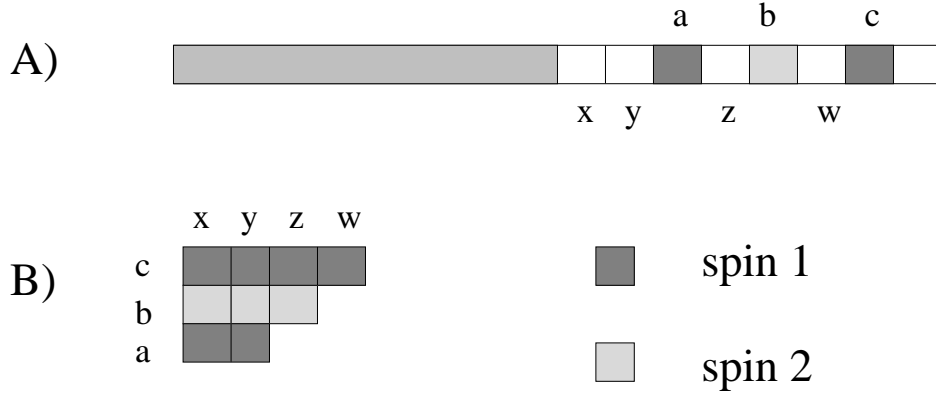

Figure 1: A example of the colored Young diagram.

We are now in position to describe the vertex operator construction of the eigenstates. In the spinless situation [20], we observed that only two types of the vertex operators, $\exp(\gamma\phi_-(\xi))$ with $\gamma = 1/\alpha, -1$, have "simple" forms after they are operated by the Hamiltonian. If we expand $e^{\phi_-(\xi)/\alpha}$ (resp. $e^{-\phi_-(\xi)}$) with respect to ξ, the coefficient of ξ^n is identified with the eigenstate for the Young diagram (n) (resp. (1^n)). Even for the system with spin degrees of freedom, we expect similar vertex operators give the eigenstates indexed by diagrams with a single row or a single column.

Let us introduce basic vertex operators,

$$\Gamma(\xi;\gamma) = \langle \mathcal{V}|e^{\gamma\phi_-(\xi^{(1)})}\cdots e^{\gamma\phi_-(\xi^{(s)})}|N\rangle = \exp\left[\gamma\sum_{\sigma=1}^{s}\sum_{n=1}^{\infty}\frac{1}{n}p_n^{(\sigma)}(\xi^{(\sigma)})^n\right]. \tag{14}$$

Then the vertex operators $\Lambda(\xi)$ and $\Omega(\xi)$ corresponding to single column and single row, respectively, are defined by

$$\Lambda(\xi) = \Gamma(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(s)}; -1) = \prod_{\sigma=1}^{s}\prod_{j=1}^{N^{(\sigma)}}(1 - x_j^{(\sigma)}\xi^{(\sigma)}), \tag{15}$$

$$\Omega(\xi) = D_\mu\Gamma(\mu, \underbrace{\xi, \cdots, \xi}_{s-1}; 1/\alpha)|_{\mu=\xi} = \frac{1}{\alpha}\sum_{n=1}^{\infty}p_n^{(1)}\xi^n\prod_{\sigma=1}^{s}\prod_{j=1}^{N^{(\sigma)}}(1 - x_j^{(\sigma)}\xi)^{-1/\alpha}, \tag{16}$$

where $D_\mu = \mu \partial/\partial\mu$. Notice that, in contrast to the spinless CSM, the derivative in (16) is essential for the case of single row. It is easily to show that

$$\mathcal{H}(x)\Lambda = \widehat{\mathcal{H}}(\xi)\Lambda, \qquad \widehat{\mathcal{H}} = -(\sum_\sigma D_{\xi^{(\sigma)}})^2 + (N+\alpha)\sum_\sigma D_{\xi^{(\sigma)}} \tag{17}$$

and

$$\mathcal{H}(x)\Omega = \widehat{\mathcal{H}}(\xi)\Omega, \qquad \widehat{\mathcal{H}} = \alpha D_\xi^2 + (N-1)D_\xi. \tag{18}$$

The derivation of (17) and (18) is straightforward. These formulae indicate that one particle and one hole excitations of the spin CSM are reduced to one-body problems in the "dual" system. Later, we will prove the general version of the formula (17).

As mentioned above, the states $\Lambda(\xi)$ and $\Omega(\xi)$ are generating functionals of the eigenstates associated with the colored Young diagram with single column (1^n) and single row (n), respectively. Namely, by expanding these states in terms of $\xi^{(\sigma)}$ (or ξ),

$$\Lambda(\xi) = \sum_{\sigma=1}^s \sum_{n_\sigma=0}^\infty J_{n_1\cdots n_s}^{(-1)}(x)(\xi^{(1)})^{n_1}\cdots(\xi^{(s)})^{n_s}, \tag{19}$$

$$\Omega(\xi) = \sum_{n=0}^\infty J_n^{(1/\alpha)}(x)\xi^n, \tag{20}$$

we obtain eigenstates $J_{n_1\cdots n_s}^{(-1)}(x)$ and $J_n^{(1/\alpha)}(x)$. Here $J_{n_1\cdots n_s}^{(-1)}(x)$ and $J_n^{(1/\alpha)}(x)$ denote the eigenstates corresponding to the single column Young diagram which has n_σ boxes with color σ and the single row Young diagram with color 1, respectively.

4 Derivation of Dual Hamiltonian

In this section, we calculate the action of the Hamiltonian on product of the vertex operators considered in the previous section. We define the vertex operator $\Lambda(x|\xi)$,

$$\Lambda(x|\xi) = \prod_{\sigma=1}^s \prod_{i=1}^{N^{(\sigma)}} \prod_{k=1}^{M^{(\sigma)}} (1 - x_i^{(\sigma)}\xi_k^{(\sigma)}), \tag{21}$$

where $M^{(\sigma)}$ denotes the number of particles with spin σ in the dual system. In what follows, we only consider the case such that $|M^{(\sigma)} - M^{(\sigma')}| \le 1$ for all σ, σ'. As in the previous section, the dual Hamiltonian $\widehat{\mathcal{H}}(\xi)$ is defined by

$$\mathcal{H}(x)\Lambda(x|\xi) = \widehat{\mathcal{H}}(\xi)\Lambda(x|\xi). \tag{22}$$

We decompose the Hamiltonian $\widehat{\mathcal{H}}$ as in (9). Then the dual Hamiltonian $\widehat{\mathcal{H}} = \sum_\sigma \widehat{\mathcal{H}}^{(\sigma)}(\xi) + \sum_{\sigma<\sigma'} \widehat{\mathcal{H}}_{int}^{(\sigma\sigma')}(\xi)$ is given by,

$$\widehat{\mathcal{H}}^{(\sigma)}(\xi) = -\sum_{k=1}^{M^{(\sigma)}} (D_{\xi_k^{(\sigma)}})^2 - \alpha \sum_{k<l} \frac{\xi_k^{(\sigma)} + \xi_l^{(\sigma)}}{\xi_k^{(\sigma)} - \xi_l^{(\sigma)}} (D_{\xi_k^{(\sigma)}} - D_{\xi_l^{(\sigma)}}), \tag{23}$$

$$\widehat{\mathcal{H}}_{int}^{(\sigma\sigma')}(\xi) = -2 \sum_{k=1}^{M^{(\sigma)}} \sum_{\ell=1}^{M^{(\sigma')}} \frac{\prod_{s(\neq\ell)}^{M^{(\sigma')}} (1 - \xi_s^{(\sigma')}/\xi_k^{(\sigma)}) \prod_{s(\neq k)}^{M^{(\sigma)}} (1 - \xi_s^{(\sigma)}/\xi_\ell^{(\sigma')})}{\prod_{s(\neq k)}^{M^{(\sigma)}} (1 - \xi_s^{(\sigma)}/\xi_k^{(\sigma)}) \prod_{s(\neq\ell)}^{M^{(\sigma')}} (1 - \xi_s^{(\sigma')}/\xi_\ell^{(\sigma')})} D_{\xi_k^{(\sigma)}} D_{\xi_\ell^{(\sigma')}}. \quad (24)$$

Here we omitted the terms which are proportional to $\sum_k D_{\xi_k^{(\sigma)}}$. Unlike the spinless CSM, the dual Hamiltonian is not similar to the original one. This fact reflects that the symmetry $\alpha \leftrightarrow 1/\alpha$ is broken.

Notice that, although this dual system does not described by the ordinary two-body interaction, its integrability is clear from our construction. Moreover, we easily see that it has the same spectrum as that of the original system. In fact, if we expand Λ,

$$\Lambda(x|\xi) = \sum_\lambda J_\lambda(x)\hat{J}_\lambda(\xi), \quad (25)$$

where $J_\lambda(x)$ is the eigenstate of the original Hamiltonian with the colored diagram $\lambda = \{\lambda_1\sigma_1, \lambda_2\sigma_2, \cdots\}$, then, because of (21), $\hat{J}_\lambda(\xi)$ should be the eigenstate of the dual Hamiltonian with the same eigenvalue.

The derivation of the dual Hamiltonian is rather lengthy. Then, for simplicity, we consider the case with two components which we denote $\{\uparrow, \downarrow\}$. Let x_i and y_i be the coordinates for the particles with up and down spin, respectively, and ξ_k and η_k be that of the dual system. The derivation of eq. (23) is straightforward. To derive eq. (24), first we observe that

$$\mathcal{H}_{int}^{(\uparrow\downarrow)}(x,y)\Lambda(x,y|\xi,\eta) = \sum_{i,j} R(x_i,y_j)Q(x_i,y_j)\Lambda(x,y|\xi,\eta), \quad (26)$$

with

$$Q(x,y) = \prod_k \frac{1}{1-x\xi_k} \prod_\ell \frac{1}{1-y\eta_\ell},$$

$$R(x,y) = \frac{x+y}{x-y} \prod_k (1-x\xi_k) \prod_\ell (1-y\eta_\ell) \left(\sum_k \frac{-x\xi_k}{1-x\xi_k} - \sum_\ell \frac{-y\eta_\ell}{1-y\eta_\ell} \right)$$

$$- 2\frac{xy}{(x-y)^2} \left(\prod_k (1-x\xi_k) \prod_\ell (1-y\eta_\ell) - \prod_k (1-y\xi_k) \prod_\ell (1-x\eta_\ell) \right).$$

Next we show that the right hand side of the expression (26) can be rewritten as the derivative with respect to ξ and η by combining following lemmas.

1. We can rewrite $Q(x,y)$ as,

$$Q(x,y) = \left(\sum_{k=1}^{M^{(\uparrow)}} A_k(\xi)\frac{1}{1-x\xi_k} \right) \left(\sum_{\ell=1}^{M^{(\downarrow)}} A_\ell(\eta)\frac{1}{1-y\eta_\ell} \right), \quad (27)$$

where $A_k(\xi) = \prod_{\ell(\neq k)} \frac{\xi_k}{\xi_k - \xi_\ell}$.

2. $R(x, y)$ is a polynomial of degree $M^{(\uparrow)}$ in x and that of degree $M^{(\downarrow)}$ in y. Namely, if we write,

$$\prod_k (1 - x\xi_k) = \sum_{n=0}^{M^{(\uparrow)}} s_n(\xi)x^n, \qquad \prod_\ell (1 - y\eta_\ell) = \sum_{m=0}^{M^{(\downarrow)}} s_m(\eta)y^m, \tag{28}$$

then $R(x, y)$ is expressed as

$$R(x, y) = \sum_{n=0}^{M^{(\uparrow)}} \sum_{m=0}^{M^{(\downarrow)}} s_n(\xi)s_m(\eta)T_{n,m}(x, y), \tag{29}$$

where

$$T_{n,m}(x, y) = \begin{cases} 0, & n = m \\ (n - m)x^n y^m + 2\sum_{r=1}^{n-m-1}(n - m - r)x^{n-r}y^{m+r}, & n > m \\ (m - n)x^n y^m + 2\sum_{r=1}^{m-n-1}(m - n - r)x^{n+r}y^{m-r}, & n < m. \end{cases} \tag{30}$$

3. For $0 \leq n \leq M^{(\uparrow)}$,

$$\sum_{k=1}^{M^{(\uparrow)}} A_k(\xi) \sum_{i=1}^{N^{(\uparrow)}} \frac{x_i^n}{1 - x_i\xi_k}\Lambda = \left(\delta_{n,0}N^{(\downarrow)} - \sum_{k=1}^{M^{(\uparrow)}} A_k(\xi)\xi_k^{-n}D_{\xi_k} \right)\Lambda \tag{31}$$

and the similar formula for y and η hold. For the derivation of this formula, we used the Euler's identity.

By the first observation, the combination on the right hand side of (26) can be expressed as derivative with respect to ξ and η by using $\partial_{\xi_k}\Lambda = \sum_i \frac{-x_i}{1 - x_i\xi_k}\Lambda$ etc. The nontriviality comes from the x, y dependence. However, from the second observation, the dependence can be reduced to their polynomial and then from the third lemma they can be replaced by the function of ξ and η. Therefore, we finally obtain the interacting part of the dual Hamiltonian $\widehat{\mathcal{H}}_{int}^{(\uparrow\downarrow)}(\xi, \eta)$.

5 Discussions and Comments

Although we know that the dual Hamiltonian we derived is integrable, many of its properties are still missing. One of such important issue is the existence of the Hermitian measure. If it exists, we can construct every eigenstate of the spin CSM as we describe in the following.

Generalizing the spinless case [20, 21], we define two transformations which map one eigenstate into another. The transformations are:

1. *Galilean transformation*: G_P

   This transformation is defined by

$$(G_P J)(x) = \left(\prod_{\sigma=1}^{s} \prod_{i=1}^{N^{(\sigma)}} x_i^{(\sigma)} \right)^P J(x). \tag{32}$$

   Since the spin CSM has the Galilean invariance, it obviously maps one eigenstate to another. At the same time, the momentum of each particle is shifted by P. On the Young diagram, G_P has an effect to attach a rectangle Young diagram (P^N) which has $N^{(\sigma)}$ rows with color σ's. This operation does not violate the rule of painting and is always possible.

2. *Integral transformation which changes the number of variables*: $\mathcal{N}(x, y)$

   Let us denote the Hermitian inner product of the original system as $\langle , \rangle_x$ and the inner product for the dual system as $\langle\langle , \rangle\rangle_\xi$. We define the integral transformation as,

$$(\mathcal{N}(x, y) J(x))(y) = \langle\langle \Lambda(y|\xi), \langle \Lambda(x|\xi), J(x) \rangle_x \rangle\rangle_\xi. \tag{33}$$

   If such inner product exists, from (21), it is clear that the Hamiltonian commutes with this operator in a following sense: $\mathcal{H}(y)\mathcal{N}(x, y) = \mathcal{N}(x, y)\mathcal{H}(x)$. Performing this transformation, we can change the number of particles for each color without touching the Young diagram.

We can construct any eigenstate of the spin CSM Hamiltonian by alternate operations of these transformations to the trivial eigenstate, namely the vacuum.

This construction of eigenstates is a straightforward generalization of the method which has been used in refs.[20, 21] to obtain the integral representation of the Jack polynomial, and indicated the remarkable identification between the Jack polynomial and the singular vectors of the Virasoro and W_N algebras. We expect that the spin CSM also possesses such an algebraic structure.

Finally, we comment on the related topics. The correspondence between the eigenstates of the spin CSM and the solutions of the Knizhnik-Zamolodchikov equation has been established [25, 26]. More recently, Felder and Varchenko [27] (see also [28]) gave some formulae for the eigenstates of the spin CSM. The Dunkl operators [29] or more precisely the representation theory of the degenerate affine Hecke algebra [31] have central rôle in the analysis of the spectrum and integrability of the spin CSM (see also [30]). It would be interesting to clarify the relation between these works and our results. Also, it is natural to consider the q-analog of our methods. The q-analog of the spin CSM has been constructed [32, 33]. We hope to turn these issue in the near future.

Appendix: Examples of Eigenstates

Here we give some explicit examples of the eigenstates of the spin CSM Hamiltonian. In the two components case, the eigenstates J_λ are written by two kinds of power sums $p_n^{(1)}$ and $p_n^{(2)}$. We distinguish between two colors of the Young diagram by using bars, for example, $J_{\lambda_1, \bar{\lambda}_2, \bar{\lambda}_3, \lambda_4, \cdots}$. The eigenstates J_λ with the Young diagrams λ up to 3 boxes are as follows:

$$
\begin{bmatrix} J_1 \\ J_{\bar{1}} \end{bmatrix} = \begin{bmatrix} p_1^{(1)} \\ p_1^{(2)} \end{bmatrix},
\qquad
\begin{bmatrix} J_2 \\ J_{\bar{2}} \\ J_{11} \\ J_{1\bar{1}} \\ J_{\bar{2}} \end{bmatrix} =
\begin{bmatrix}
\alpha & 0 & 1 & 1 & 0 \\
0 & \alpha & 0 & 1 & 1 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
p_2^{(1)} \\
p_2^{(2)} \\
p_1^{(1)} p_1^{(1)} \\
p_1^{(1)} p_1^{(2)} \\
p_1^{(2)} p_1^{(2)}
\end{bmatrix},
$$

$$
\begin{bmatrix}
J_3 \\ J_{\bar{3}} \\ J_{21} \\ J_{2\bar{1}} \\ J_{\bar{2}1} \\ J_{\bar{2}\bar{1}} \\ J_{111} \\ J_{11\bar{1}} \\ J_{1\bar{1}\bar{1}} \\ J_{\bar{1}\bar{1}\bar{1}}
\end{bmatrix} =
\begin{bmatrix}
2\alpha^2 & 0 & 3\alpha & 2\alpha & \alpha & 0 & 1 & 2 & 1 & 0 \\
0 & 2\alpha^2 & 0 & \alpha & 2\alpha & 3\alpha & 0 & 1 & 2 & 1 \\
-\alpha & 0 & \alpha-1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & \alpha-1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & \alpha-1 & 0 & 0 & 1 & 1 & 0 \\
0 & -\alpha & 0 & 0 & -1 & \alpha-1 & 0 & 0 & 1 & 1 \\
2 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p_3^{(1)} \\
p_3^{(2)} \\
p_2^{(1)} p_1^{(1)} \\
p_2^{(1)} p_1^{(2)} \\
p_2^{(2)} p_1^{(1)} \\
p_2^{(2)} p_1^{(2)} \\
p_1^{(1)} p_1^{(1)} p_1^{(1)} \\
p_1^{(1)} p_1^{(1)} p_1^{(2)} \\
p_1^{(1)} p_1^{(2)} p_1^{(2)} \\
p_1^{(2)} p_1^{(2)} p_1^{(2)}
\end{bmatrix}.
$$

Next we show some examples of the eigenstates of N variables in the coordinate space. The eigenstate and its eigenvalue are parameterized by a non-negative sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N \geq 0)$ and its set $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$, respectively. We define a monomial $m_\lambda \equiv \prod_{i=1}^{N} x_i^{\lambda_i}$. The first few examples of the eigenstates J_λ, in the case of $\sum_{i=1}^{N} \lambda_i = N$, are as follows:

$$J_1 = m_1, \qquad J_{20} = (\alpha+1)m_{20} + m_{11}, \qquad J_{11} = m_{11},$$

$$J_{300} = (\alpha+1)(2\alpha+1)m_{300} + (\alpha+1)\left(2m_{210} + 2m_{201} + m_{120} + m_{102}\right) + 2m_{111},$$

$$J_{210} = (\alpha+2)m_{210} + m_{111}, \qquad J_{111} = m_{111}.$$

Eigenstates J_{02}, J_{030} and J_{201} etc. have the similar forms. Remark that if we sum up J_λ over λ's which have the same set $\{\lambda\}$, then we obtain the Jack polynomial with the corresponding Young diagram $\{\lambda\}$.

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